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KLEIN'S DOUBLE DISCONTINUITY REVISITED:
CONTEMPORARY CHALLENGES FOR UNIVERSITIES
PREPARING TEACHERS TO TEACH CALCULUS

Carl Winsløw*, Niels Grønbaek**

Abstract – Much effort and research has been invested into understanding and bridging the ‘gaps’ which many students experience in terms of contents and expectations as they begin university studies with a heavy component of mathematics, typically in the form of calculus courses. We have several studies of bridging measures, success rates and many other aspects of these “entrance transition” problems. In this paper, we consider the inverse transition, experienced by university students as they revisit core parts of high school mathematics (in particular, calculus) after completing the undergraduate mathematics courses which are mandatory to become a high school teacher of mathematics. To what extent does the “advanced” experience enable them to approach the high school calculus in a deeper and more autonomous way? To what extent can “capstone” courses support such an approach? How could it be hindered by deficiencies in the students’ “advanced” experience? We present a theoretical framework, based on the anthropological theory of the didactic, for an analysis of these questions, as well as a number of critical observations and reflections on how these questions appear as challenges in the Danish institutional context.

Key words: university mathematics, calculus, teacher education, praxeology

RETOUR SUR LA DOUBLE DISCONTINUITÉ DE KLEIN : QUEL
USAGE DES MATHÉMATIQUES UNIVERSITAIRES POUR
L’ANALYSE AU LYCÉE ?

Resumé – Beaucoup d’efforts et de recherches ont été consacrés à comprendre et à combler les « lacunes » que beaucoup d’étudiants vivent face aux contenus et aux attentes qu’ils rencontrent au début d’études universitaires avec un composant lourd de mathématiques, souvent en contexte de cours d’analyse. Nous avons plusieurs études des dispositifs de transition, des taux de réussite et d’autres aspects de ces problèmes “de transition à l’entrée”. Dans cet article, nous considérons la transition inverse, vécue par les étudiants de l’université quand ils revisitent les éléments des mathématiques du secondaire (en particulier, l’analyse) après avoir suivi les cours de mathématiques de licence obligatoires pour devenir professeur de mathématiques au secondaire. Dans quelle mesure l’expérience de mathématiques « avancées » leur permet-elle

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d'aborder l'analyse du lycée d'une manière plus profonde et plus autonome? Dans quelle mesure des cours de type "cap stone" (clé de voûte) peuvent-ils soutenir une telle approche? Quelles contraintes résultent des lacunes dans l'expérience de mathématiques «avancée» des étudiants ? Nous présentons un cadre théorique, fondé sur la théorie anthropologique du didactique pour l'analyse de ces questions, ainsi qu'un certain nombre d'observations et de réflexions critiques sur la façon dont ces questions se présentent comme des « challenges » dans le contexte institutionnel du Danemark.

Mots clés: mathématiques universitaires, analyse, formation des enseignants, praxeologie

VOLVER EN LA INTERRUPCIÓN DE DOBLE KLEIN: ¿QUÉ USO DE MATEMÁTICAS DE LA UNIVERSIDAD DE ANÁLISIS EN LA ESCUELA?

Gran cantidad de esfuerzo y la investigación se ha dedicado a entender y llenar los "huecos" que muchos estudiantes enfrentan en su contenido y de las expectativas que se enfrentan en el inicio de los estudios universitarios con un componente fuerte de las matemáticas, a menudo siendo analizado contexto. Tenemos varios estudios sobre medidas transitorias, las tasas de éxito y otros aspectos de estos problemas "de transición en la entrada." En este artículo, se considera la transición inversa experimentado por los estudiantes universitarios, cuando vuelven a visitar los elementos de la matemática escolar (en particular, el análisis) después de completar los cálculos de la licencia se requiere para convertirse en profesor de matemáticas en el nivel secundario. ¿En qué medida la experiencia de matemáticas "avanzado" le permite abordar el análisis de la escuela de la más profunda y de forma independiente? ¿En qué medida los cursos de tipo " toque final " pueden apoyar tal enfoque? ¿Qué limitaciones resultantes brechas en experiencia "avanzado" estudiantes? Se presenta un marco teórico basado en la teoría antropológica de la didáctica para el análisis de estas cuestiones, y una serie de observaciones y reflexiones críticas sobre cómo estas preguntas aparecen como desafíos en el contexto institucional de Dinamarca.

Palabras-claves: matemáticas académicas, el análisis, la formación del profesorado, praxeología

1. A CLASSICAL PROBLEM REVISITED

In an often-quoted preface, Klein (1908/1932, p. 1) observed that students face a “double discontinuity” as they move from high school¹ to university, then back again to a career as school teachers:

The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular [*sic*], the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.

The first ‘discontinuity’ concerns the well-known problems of transition which students face as they enter university, a main theme in research on university mathematics education (see e.g. Gueudet, 2008). The second ‘discontinuity’ concerns those who return to school as teachers and the (difficult) transposition of *academic knowledge gained at university* into *relevant knowledge for a teacher*.

Since Klein's days, and particularly in the past few decades, much research has been devoted to mathematics teacher knowledge, especially with regard to the contributions of initial teacher education (see e.g. Ball, Hill & Bass, 2005; Evens & Ball 2009; Buchholtz, Leung, Ding, Kaiser, Park & Schwartz., 2013). The vast majority of these studies focus on the teachers' preparation and knowledge about primary and lower secondary level mathematics, where the distance to university level mathematics is clear. In this paper our focus will be on future teachers of calculus at the high school (upper secondary) level, in particular on how an academically oriented bachelor programme in pure mathematics may (or may not) contribute knowledge which is relevant to the task of teaching high school calculus. Focusing on this level, a smaller gap between university mathematics and the mathematics to be taught by the teacher is often

¹ In this paper, we consistently use the term “high school” to translate the Danish term *gymnasium*, a national school institution at grade level 10-12 which prepare its students for higher education - as do similar institutions in other countries (*lycée* in France, *Gymnasium* in Germany, and to some degree *high schools* in the United States).

assumed. Indeed it is common for initial education of teachers at this level to include a substantial amount of pure mathematics courses from a university, which is only natural if we agree with Klein that

...the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them. (Klein, 1908/1932, p. 192)

But even on this premise, there is no reason to expect an automatic transfer (or “trickle-down theory”, in the terms of Wu, 2011) between the advanced mathematical knowledge gained at university and the tasks of teaching calculus in a high school.

Here are some of the reasons why the preparation of teachers is particularly interesting to us in the specific case of calculus:

- In high school, calculus is one of the most advanced topics, and it is usually taught in a quite informal style, leaving the teacher with delicate choices and tasks of explanation (for example in relation to notions of limit, as studied by Barbé et al., 2005);
- The “cliffs and whirlpools” of this topic can indeed be usefully approached using elements of typical undergraduate courses in pure mathematics, particularly in real analysis and algebra, but it could still be necessary to learn such an approach explicitly at university;
- Calculus is particularly affected by the increasing use of symbolic calculation devices in high schools in many countries, and this use in itself leads to important challenges for teaching (see e.g. Guin, Ruthven, Trouche, 2005).

As a result, Klein’s problem has significant and critical aspects for the specific case of high school teachers and calculus in our time.

Klein’s own *answer* to the problem was that indeed, *university instruction* [must take into account] *the needs of the school teacher* (ibid., p. 1) His proposition for so doing was a series of lectures specifically designed to *help them see the mutual connection between problems in the various fields (...) and more especially to emphasize the relation of these problems to those of school mathematics* (ibid., p. 1-2). He emphasizes that to follow these lectures, students should already be *acquainted with the main features of the chief fields of mathematics* (ibid. p. 1), as a result of their previous university studies. In essence, he thus advocates the introduction of what is known today as *capstone courses* for mathematics teachers. In this paper, the term ‘capstone course’ is used to indicate a study unit which is located towards the end of an academic study program, with the aim of concluding or ‘crowning’ the experience, and to link academic competence and training with the needs of a professional

occupation (cf. Durel, 1993; Winsor, 2009). We note here that whilst the *term* is common mainly in North America, the same kind of courses also exists, with variations, in other parts of the world.

Klein's proposition implies two concrete questions: how could a capstone course detect and remedy gaps between students' knowledge and relevant knowledge for the teacher, and what is the appropriate 'higher standpoint' required before this course?

To approach and sharpen these questions, we present (in Section 2) a reformulation of Klein's "double discontinuity", based on a framework inspired by the anthropological theory of the didactical. In Section 3, we explain the method and context that we choose to study these questions, namely observing students enrolled in a capstone course within a mainstream university programme in pure mathematics, at the University of Copenhagen. This leads to identifying a number of key "challenges" arising for such a course in the setting of a contemporary academic mathematics programme and which are presented in Section 4. In the concluding Section 5, we expose wider implications and perspectives for research and reflect on how our paper is situated in, and contributes to, the existing literature.

Before entering into the main part of the paper, we specify our topic and goals relatively to previous research. The topic is certainly related to both mathematics teacher education and secondary level mathematics education, but our study focuses on a specific problem for university mathematics education, namely that of using students' "advanced" mathematical experience to gain a deeper insight into calculus at high school level. Our paper contributes to the existing literature on this problem by providing a new theoretical model for the transition to be achieved in students' relationship to mathematics, and by showing how the model can be used to analyse students' work in a capstone course where the "advanced" experience is a bachelor programme in pure mathematics (as found in many universities). The fact that such an experience does not automatically ensure "deep" knowledge of elementary mathematics is not new. For instance, Buchholtz et al. (2013) presented a systematic study of student knowledge of similar kind in Germany, China, Hong Kong and South Korea, using a (mainly quantitative) test design. That study, based on a diagnostic test, gives a global picture of the shortcomings, which motivate capstone courses. Our study aims to identify specific challenges which students meet in more complex tasks from a course set up specifically to explore high school calculus while drawing on their "advanced" experience. It should be noted here that the similar problem for algebra has been recently addressed, in the form of a textbook, by Cuoco and Rotman (2013). There are several other

descriptions, resources and models of capstone courses for future teachers (e.g. Winsor, 2009), but our approach to analyse students' work and the results found in the calculus context are certainly new. As pointed out above, we consider calculus to be of particular interest relatively to capstone courses for future high school teachers because this is a new and difficult domain for students in high school, and further university courses in analysis could be expected to support students' return to the more elementary calculus topics taught in high school.

2. THEORETICAL FRAMEWORK

We find it useful to relate Klein's "double discontinuity" to the following three dimensions which call for separate attention, although they are not independent:

- The institutional context (of university vs. school)
- The difference in *the subject's role* within the institution (a student in university or school, vs. a teacher of school mathematics)
- The difference in *mathematical contents* (elementary vs. advanced)

In his book, Klein mainly focuses on the latter dimension, and the solutions he proposes can be described as "building bridges" at the level of contents (sometimes with explicit advice to teachers on how to expose a subject). Some of Klein's general proposals have been implemented in the course of the twentieth century. For instance, functions and calculus have become a central part of upper secondary education in most European countries. Klein considers the institutional dimension only in the introductory remarks, where he actually points out some basic and problematic discrepancies between basic aims of the two institutions:

For a long time ... university men were concerned exclusively with their sciences, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics. (*ibid.*, p. 1).

One can safely say that this problem, which Klein describes in past tense in 1908, is not less important today (see e.g. Cuban, 1999). In fact, the causes for university mathematicians' lack of concern and contacts with school mathematics have increased as a consequence of the evolution of institutions: the workforce and institutional frameworks of mathematical research have expanded tremendously since the days of Klein, and mathematics programs and courses

prepare students for a still wider range of professions. At the same time, national institutions of high school have developed in all Western countries, with a much larger audience and a much wider scope, which are no longer limited to preparing elite students to university education.

To model Klein's problem, with the three dimensions stated above, we make use of central tools from the anthropological theory of the didactical (ATD), initiated by Chevallard (1991, 1999, 2002). Our main justification for this choice of theoretical framework is that it provides tools for analysing the interplay between human practices and institutions and the ways in which the former are conditioned and constrained by the latter, through the relations to practices imposed or encouraged by institutions.

In ATD, human knowledge and practice are modelled as praxeologies, which are highly structured organisations of practice blocks and theory blocks (Chevallard, 1999). A practice block consists of a type of tasks and a corresponding technique, which can be used to accomplish the tasks of the given type. A theory block is attached to a family of practice blocks and consists of technology, i.e. discourse about the techniques, and theory, which justifies, explains and unifies one or (typically) more technologies. Readers unfamiliar with these notions are invited to consult Barbé et al. (2005, sec. 2) or similar references where they are exposed in more detail. We just emphasize here that while techniques and types of task correspond to each other, a theory block serves to explain, distinguish, unify and justify a (potentially large) collection of techniques.

An institution is, roughly speaking, a collection of people who share a collection of praxeologies. An example could be everyone involved in the Danish high school, with its repertoire of teaching and study practices in a variety of disciplines and modes. The example also shows why we said "roughly speaking": an institution certainly involves a number of concrete people, with a variety of positions or roles relative to the praxeologies (in the example, being a student or a teacher); but these people come and go over time, and so it is more correct to say that it is the main types of positions which make up the institutions. The praxeologies of an institution also evolve over time, but the positions of its members (e.g., teachers and students) normally remain so stable that we can nevertheless continue to speak of the same institution. We notice also that people in the same position (e.g., students) may certainly develop somewhat different relationships to the praxeologies in which they take part, as a function of their position in the institution; still, we may wish to identify and study a smaller

number of typical relationships, as does Klein in the famous quote given in section 1.

The notation $R_I(x,o)$ was introduced by Chevallard (1991) to indicate the relationship of a position x (within an institution I) to a praxeology o . The notation is just a compact abbreviation, not a mathematical formula. It enables us to represent, in a compact way, the three dimensions of the “double discontinuity” of Klein, as the two passages (each indicated by an arrow):

$$R_{HS}(s,o) \rightarrow R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$$

where: o indicates a mathematical praxeology worked on in high school (HS) by teachers (t) and students (s), while ω indicates a mathematical praxeology, which the students (σ) encounter at university (U). We stress that s , σ and t do not designate people as such, as the same person can be consecutively in these positions within a few years. The arrows above, on the other hand, refer to this passage for a single person, occupying the positions and undertaking the relationships indicated.

In particular, the second part of Klein’s problem consists in the lack of (perceived) relevance of $R_U(\sigma,\omega)$ to $R_{HS}(t,o)$, even for the case where o is similar or perhaps a part of ω . While there are certainly many mathematical praxeologies which the student has to relate to at university, but are not close to anything taught in high school, most of the mathematical praxeologies taught in high school (e.g. the practices and knowledge related to derivatives) find some counterpart at university level, often with a more theoretical, general etc. stance (for instance, differentiation is also considered for several variables, complex functions etc.). To identify and exploit these counterparts is one strategy to tackle the second part of the transition.

In terms of the above model, a capstone course for future teachers (as defined in the introduction) aims to develop relationships of type $R_U(\sigma,o)$ while drawing on $R_U(\sigma,\omega)$, and in view of the needs for a future relationship of type $R_{HS}(t,o)$. As a capstone course takes place within the university programme, the school as an institution remains distant at least in contexts where no pre-graduation teaching practice is organised; but it is clear that the motivation of the course is to achieve relationships to school mathematics, which, from the course organisers point of view, will be useful once the student becomes a teacher. A main task of our study is to elucidate how the success and outcome of this endeavour can differ according to the qualities of $R_U(\sigma,\omega)$. In some cases, we will notice that a further development $R_U^*(\sigma,\omega)$ of $R_U(\sigma,\omega)$ is needed or at least advantageous to achieve a

satisfactory result $R_U(\sigma, o)$; in such a case, the complete course of the students may be described as

$$R_U(\sigma, \omega) \rightarrow R_U^*(\sigma, \omega) \rightarrow R_U(\sigma, o).$$

Now, our overall research questions can be formulated as follows :

RQ1. What kinds of (new) relationships $R_U(\sigma, o)$ are useful to build between university students and mathematical praxeologies from high school, within a university programme in mathematics? – here “useful” is to be understood as arguably useful for the aim of preparing students for secondary school teaching.

RQ2. What are the main obstacles to building these relationships, in terms of the relationships $R_U(\sigma, \omega)$ to mathematical praxeologies which student already have?

RQ3. What further developments $R_U^*(\sigma, \omega)$ of the students' relationships to university mathematical praxeologies may be desirable or necessary in order to achieve the goals identified as answers to RQ1?

For us RQ2 is central and then it will be the most intensively addressed in this paper.

We will adopt a relatively modest interpretation of RQ1: “kinds of relationship” will be interpreted as students' capability to solve tasks coming from o itself or from immediately related mathematical praxeologies ω , typically with ω -tasks being about developing theory blocks of o (for instance, proving a result which in high school was implicitly or explicitly assumed). Then the usefulness of students' capability of solving these tasks will be argued through their direct relevance to solve didactic tasks (in the sense defined by Chevallard, 2002, p. 5) in high school (HS).

We will then seek answers to RQ2 and RQ3 as they pertain to concrete tasks with which students in a capstone course encounter some or many difficulties, especially when their relationships to university mathematics hinder or enhance their opportunities of solving the types of tasks identified in RQ1. In fact, the answers to RQ2 and RQ3 should then identify concrete challenges for university programs in terms of insufficient or desirable relationships $R_U(\sigma, \omega)$ and desirable, more advanced relationships $R_U^*(\sigma, \omega)$ – where the meaning of “desirable” could relate to the topos of students both as regards the technical and theoretical levels of ω .

It is a common experience that university students often deal with “advanced” praxeologies ω in less than advanced ways, that is with a main focus on handling tasks using techniques taught in the course, and with a mostly passive relationship to the theory blocks (Winsløw, 2008). A main point in our earlier work (Grønbaek and Winsløw,

2007) on didactical engineering for the teaching of real analysis can be described as constructing new formats of student work which enable the students to develop a stronger didactic autonomy (Chevallard, 2002, p. 9) in relation to the theory blocks of advanced mathematical praxeologies.

3. CONTEXT AND METHODOLOGY

The research questions presented above can only be treated meaningfully through patient observation and reflection on cases. Our empirical cases, presented in the next section, come from three years of observation done in the capstone course *UvMat* (an abbreviation for the Danish equivalent of *Mathematics in a Teaching Context*) at the University of Copenhagen.

The course *UvMat* caters to students in the third year of the B.Sc. programme in mathematics and aims to help students relate relevant parts of their academic bachelor courses to high school mathematics, in view of a professional life as teachers. In short, it is a capstone course as defined above, and is based on more or less well founded practical hypotheses related to the research questions formulated above. Such hypotheses can be *tested* but also *generated* from observing and analysing students' work in the course, which is the main idea of our methodology.

Our context

To become tenured as such in Denmark the candidate must have a Master degree that meet certain content matter requirements specified by the Ministry of Education; the only further requirement is to take certain courses on pedagogy during the first two years of teaching. The latter courses do not address content matter, which is only studied at university, prior to practice. Study programs in mathematics mostly focus on scholarly progress – which is consistent with Klein's vision of teachers considering elementary mathematics from a higher standpoint.

UvMat is not mandatory, even for prospective high school teachers. It has between 15 and 30 students each year. Mostly participants do only a minor in mathematics (about two years of the bachelor programme) along with a major in another subject. The failure rate at the final exam is relatively high (15-25%, depending on the year).

The overall course goal of *UvMat* is to enable the students to work with subjects of high school mathematics from the higher standpoint

of their present mathematical knowledge, that is, to develop $R_U(\sigma, o)$ based on $R_U(\sigma, \omega)$. The official course goals are for students to become capable of (1) solving more demanding problems within high school mathematics, (2) formulating simple as well as challenging and problem-based questions, (3) relating critically to relevant resources (such as text books, technological tools, and websites), and (4) working with applications of mathematics in other subjects.

Klein's (1908) lectures address the transition $R_U(\sigma, \omega) \rightarrow R_U(\sigma, o)$ almost exclusively at the *theoretical* level of the praxeologies, with little emphasis on techniques for problem solving, formulation of tasks etc. The aims of UvMat, in contrast, also contain a strong emphasis on working with *practical blocks* from relevant high school praxeologies.

Methodology

In order to identify concrete and major qualitative aspects of the transition $R_U(\sigma, \omega) \rightarrow R_U(\sigma, o)$, we mainly analysed students' individual and written work on specific *o*-tasks set during the course. Specifically, our main data were students' responses to weekly assignments and final exam problems. The written assignments consisted of seven weekly problem sets, six to be answered by groups of 1-4 students and one to be answered individually. The final exam was also individual.

The analysis of student praxeologies, as reflected in their written work, was carried out by creating a coding scheme for each item, based on a reference model for the mathematical praxeology in question. Each item was subdivided into minimal subtasks and we then identified the techniques each student had employed (or not employed), as a path to identify the exact nature of their difficulties. A detailed example and more explanation of the method are provided in the appendix. In the next section, we present a number of striking cases from this type of analysis.

In addition to this, we have made informal (but very limited) use of the following types of data: (1) email correspondence with some of the students, to support our interpretation of certain problematic answers; (2) a focus group interview of three experienced high school mathematics teachers, to gauge the relevance of the exam problems to authentic teaching of actual secondary mathematics; (3) a voluntary test based on two of the exam questions, given to a voluntary sample of 23 third year students not enrolled in UvMat and analysed with the same method as mentioned above; this will be briefly referred to in our description of main challenges, in view of assessing the extent to which they come from more general shortcomings of students'

relationships to relevant praxeologies taught in the university programme.

4. RESULTS: MAIN CHALLENGES FOR CAPSTONE COURSES

We now present what we consider main observations from our own with UvMat, organised according to the complexity of the mathematical praxeologies which students work with in that course in order to develop their relationship to praxeologies of type ω and o . This complexity varies from students' capacity to use and explain standard algebraic techniques and technology, to autonomous research and study on a theoretical calculus topic in view of presenting it at a given (high school) level. This leads us to expose five major challenges, which are not just challenges for a single course but also for an entire university programme that is supposed to provide its students with adequate mathematical preparation for teaching calculus at high school. Each of these challenges represent partial answers to our research questions, as they represent concrete potentials and obstacles for building a relationship of type $R_U(\sigma, o)$ which are related to the actual relationship of type $R_U(\sigma, \omega)$ and which, in some cases, suggest the need for developing $R_U(\sigma, \omega)$ further.

We notice here that while many students certainly succeeded with a number of tasks and challenges in the course, we focus here on the most problematic and difficult challenges identified from the analyses described above.

Challenge 1: autonomous control of algebraic reasoning

We begin with a challenge which did not occur to us as important during the first years of teaching the course, but which emerged as a serious surprise during the most recent edition (2012). It concerns students' mastery and control of basic algebraic reasoning – that is, not just manipulating symbolic expressions, but using such manipulation in a correct and transparent way. A simple example, to be considered in the following, is to solve an equation with one unknown by hand, with full control of the meaning of all steps.

We speculate that extensive CAS-use by students could be one reason why this is increasingly a challenge even for students 2 or 3 years into a pure mathematics programme. While computer algebra systems (CAS) can certainly solve many types of algebraic tasks with great precision and speed, the user of such programmes (whether teacher or student) needs to be able to explain and (at least in simple

cases) control both input and output in mathematical terms. For these and other reasons we maintain that university students should have a strong and autonomous command of algebraic techniques and technology, including a capacity to develop valid and clear reasoning involving algebraic operations such as the reasoning involved in solving a variety of algebraic equations.

One may also argue that for a teacher at upper secondary school, it is particularly important to be able to formulate algebraic reasoning in a variety of ways and settings, including the logical subtleties involved in solving equations like

$$\sqrt{2x+12}-1=x$$

where a mechanical step-by-step rewriting must be supplied with a firm control of the implications between various forms (for the detailed analysis, see the Appendix). A teacher is also expected to be able to identify and explain the challenges which her students may face with the task, and to describe the challenge in more general terms involving, for instance, mathematical technology and theory related to implications and (solution) sets.

In the 2012 version of our course, we realized from the beginning that the students' relationship to algebra and functions was partly insufficient even at the technical level and, to a larger extent, at the level of technology and theory – for instance, about the meaning of the solution to an equation.

At the final exam, students were asked to provide a detailed solution to the equation above and then to identify challenges in the task that could be critical for (high school) students. Here, 4 out of 13 students gave a wrong solution to the equation, and were thus certainly unable to answer the second part. This level of failure on a high school level task was a real surprise to us; to investigate the phenomenon further, we gave the basic task of solving the equation to an informal sample of 23 other students in their last year of bachelor studies². Among them we found even higher rates of failure, both at the technical and technological level (i.e. both as regards the practical means employed to solve the task, and the explanations and justifications offered by the students - we stress that in this exercise, the latter were explicitly emphasised as important, and also that a satisfactory technology involves attention to the meaning of implications, as explained in the appendix). This confirms that a significant number of students who take, or could take UvMat, in fact

² The bachelor (B.Sc.) degree in Denmark takes three years and corresponds roughly to the *licence* in a French university.

have a relationship to algebraic reasoning which is insufficient for developing and explaining the solution of simple equations.

Throughout the course we encountered many other instances of students' inadequate relationships to precalculus algebraic techniques and technology, and this certainly constitutes – as a partial answer to RQ2 – a real obstacle to building more advanced relationships to praxeologies involving algebraic techniques, which are prerequisites to most calculus praxeologies.

While one could put the blame on the high school (thus on the final state of $R_{HS}(s,o)$), the university cannot be excused for leaving students' relationship to algebra in the same state. Our experience suggests that teachers of capstone courses must be prepared to detect and deal with this kind of clearly inadequate state of $R_U(\sigma,\omega)$, which in fact calls for a remedial course rather than a capstone course, and also these teachers must engage in dialogue with teachers of “main” bachelor courses (e.g. in advanced calculus and linear algebra) regarding the appropriate timing for tackling such problems with basic technical and technological capacity.

Challenge 2: non-standard use of standard calculus techniques

With the previous challenge in mind, we now approach students' relation to basic techniques from high school calculus, which include calculations and uses of derivatives as in the following exercise, which also appeared in the final exam of our course and where the technical level is only slightly above what is required in Danish high school (especially techniques involve derived from the mean value theorem, such as the use of the sign of the derivative to determine monotonicity):

Assume that the function f is a solution to the differential equation

$$\frac{dy}{dx} = \exp(y^3).$$

- Show that f is strictly increasing.
- Show that f is twice differentiable and find an expression for f'' .

[Note of the authors: here, we literally translate the Danish formulation of the exercise.]

The main challenge in a) is to notice and use that the derivative of f is strictly positive. Question b) involves a somewhat challenging use of the chain rule on $\exp(y^3)$, where the challenge is that this expression should be derived with respect to a free variable in $y = y(x)$ which is implicit. A theoretical justification for the existence of the second derivative can be given from the fact that, according to the given

assumptions, dy/dx is a composition of differentiable functions; in fact a main difference between high school and university level treatments of the chain rule is to explicitly state and prove that a composition of differentiable functions is again differentiable.

While solving question a), an obstacle appears for many of our students to arise from their previous experience with differential equations: they are used to *solve* the equation before (perhaps) considering the solutions. In this case, their standard technique (separation of variables) leads to an integral which cannot be computed in closed form – indeed, a few boldly take this route and get stuck, while almost half of the students don't answer or state they “can't solve it” or the like. As a result, about half of students did not produce a correct answer to this question. The “solving reflex” is clearly counterproductive.

At the exam, 22% of the 17 students were able to do the computational part of question b) (“find an expression..”), and in the test with a control group of 23 students at the same level (but who did not attend the course) the rate of success was slightly lower. These relative failures may also reflect, at least in part, the “solving reflex” obstacle discussed above. As regards the first part of question b), on the existence of the second derivative, no student ever addresses this beyond (trying) to compute it.

Challenge 3: autonomous use of instrumented calculus techniques

In Danish high school, calculus currently involves massive use of computer algebra systems (CAS) such as *Maple* or *TI-Nspire*. In terms of praxeologies, these devices offer techniques (called *instrumented techniques* in the literature, cf. e.g. Lagrange, 2005) that allow the user to solve tasks such as equations, computing limits, plotting graphs and so on. This means that high school praxeologies include instrumented techniques not just as options, but also as students' preferred or (more rarely) unique techniques for many types of tasks. In the mathematics programme of the University of Copenhagen (U), calculus praxeologies ω involve the use of *Maple* (a professional CAS) in the first semester, but the main goal for $R_U(\sigma, \omega)$ in the programme is on developing a closer and more precise relation to the theoretical level (see also Gyöngyösi, Solovej & Winsløw, 2011). This includes, for instance, appropriate use of precise definitions, providing details or explanations of proofs etc. As a result, instrumented techniques are much less dominant in students' calculus practices at university.

In the course, we revisit high school level praxeologies to explore the effects and potentials of instrumented techniques, while making use of the theoretical knowledge obtained at *U*. The transition

$R_U(\sigma, \omega) \rightarrow R_U(\sigma, o)$ aimed for involves at least two parts: (1) a more subtle use of instrumented techniques along with non-instrumented ones in order to explore difficult tasks and discuss their theoretical perspectives; and (2) to develop the comprehensive and explicit knowledge of instrumented techniques which is needed to design tasks within an instrumented learning environment, and to explain and assess results.

The first part is most directly based on $R_U(\sigma, \omega)$ and we will concentrate on it here. The more delicate relation to instrumented techniques, which we aim for, is better learned at university. In particular, students should learn to combine instrumented techniques with non-instrumented ones, and achieve a better balance between the following aspects of CAS-use:

- The *technical use*, where the instrumented technique appear mainly as an easy way to get certain tasks done (this is the main role of CAS to high school students, who sometimes view all tasks of a given type as *either* “very easy”, when the instrumented technique works, *or* “impossible”, when it doesn’t work - i.e. they are “push button techniques” in the sense of Lagrange, 2005);
- A *technological use*, to explain and present results using CAS (for instance, to produce illustrative graphs or tables; this occurs more rarely in high school and university, although pc-based CAS-tools have increased the ease of integration of CAS output with normal text);
- A *theoretical use*, such as using CAS as an experimental tool, to investigate a more abstract problem, typically with instrumented techniques being used as complements to pen-and-paper techniques (this occurs in university albeit rarely, see for instance Gyöngyösi et al., 2011).

This more balanced use of CAS is an important example of the new relationships $R_U^*(\sigma, \omega)$ at which we aim, especially in the context of praxeologies ω in which students have little or no experience with instrumented techniques.

Investigating rational functions

As an example of the difficulties this aim meet with, we consider an item extracted from the weekly assignments:

Maple gives $a = 1.414213562$ as the 10-digits decimal expansion of $\sqrt{2}$.

Investigate the functions $f(x) = \frac{x^2 - 2}{x - a}$ and $g(x) = \frac{x^2 - 2}{x - \sqrt{2}}$

numerically, algebraically and geometrically. Explain essential differences between the two functions.

A university praxeology ω generated by this task may involve techniques and technology related to poles, removable singularities, polynomial division, and density of \mathbb{Q} in \mathbb{R} . The analogous high school praxeology o involves, in the Danish context, techniques and technology related to vertical asymptotes, zooming in on plots, and numerical tables. The first part of the item aims to supply a basis for relating the two.

In the geometric investigation most students used *Maple* to give plots showing that the graph of f has a vertical asymptote whereas the graph of g is linear. Some students demonstrated the need to zoom in, by giving plots where “graphs appear identical” and plots where “graphs are clearly different” (students’ wording), and one group noted that the necessary degree of zooming is related to the accuracy of the decimal expansion. This was mostly satisfactory.

All students did the numerical investigations using tables of function values. They are generally more unfocused than the geometric investigations or even off the point (for instance using the values $x = 1, 2, \dots, 10$). One student comments on his table: “it is difficult to see that f is unbounded”. But other students produced tables of function values that clearly suggest this. Several students interpreted “numerical” plainly as a table in itself, mixing exact and floating-point numbers.

Instrumented techniques were not used in students’ algebraic analysis of the functions, and while that analysis was adequate, the students missed opportunities (not least for future teachers) related to algebraic CAS-techniques and to coordination with the geometric and numeric investigations based on CAS.

Modelling with differential equations

Differential equations represent another calculus topic where we meet the potential and need for all three aspects of CAS use. In one UvMat assignment, we focused on autonomous differential equations, exemplified by fish catch models

$$\frac{dN}{dt} = kN(K - N) - F \quad (\text{FCM})$$

where N denotes the population size at time t , the constants k and K are model parameters to be interpreted, and F is catch per time unit. Investigating such models qualitatively (using CAS) clearly requires one to go beyond the technical use in which one seeks solutions in closed form; in fact, the infinity of closed form solutions to (FCM) may generally say very little about the properties of the model. Instead, one can use CAS to explore (FCM) by producing *phase*

diagrams and *direction fields* as well as concrete *solution curves*, and then discuss the relation between the model (FCM) and these three types of diagram. The role of instrumented techniques consequently differs from the example above, as they become essential to the theory blocks, and will be so also in a high school learning environment. We denote by o the corresponding praxeologies, which form part of the organisations commonly taught in the most advanced mathematics course in Danish high school (taken by about 40% of its students), and where instrumented techniques play a major role.

To produce and make use of the three types of diagram requires new instrumented techniques. These were introduced in a lecture based on an interactive *Maple* sheet with integrated mathematical text, recalling also basic knowledge about differential equations and more basic instrumented techniques such as numerical and symbolic solution commands. In the subsequent project assignment, the students worked with two special cases of the equation (FCM).

The target relationship $R_U(\sigma, o)$ requires that the three types of diagrams are *interpreted* and *related* to one another, as well as to the differential equation. The general challenge concerning students' productive of coherent, reasoning technology (already present in Challenge 1) is accentuated through the use of instrumented techniques.

As an example of observation that illustrate these challenges, we mention a relatively high-performing student who had produced the plot of solution curves on top of a direction field (shown in Fig. 1) to illustrate typical behaviour with respect to equilibrium states. This requires appropriate choices of initial conditions and selections of the plot dimensions.

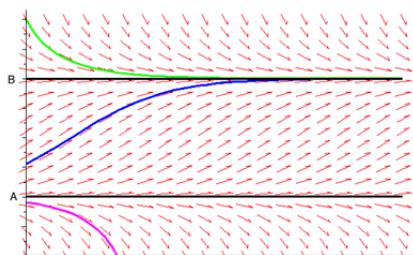


Figure 1. Direction field.

Later in the text the student makes a typing error in *Maple* and therefore gets a wrong form ($N(t) = 1 + ce^{-2t}$) of the complete

solution; but he does not notice that this solution set is qualitatively very different from the set of solution curves in Fig. 1 (for instance, the numbers of constant solutions differ). The challenge of coordination and integration between CAS-output and non-instrumented techniques, is very visible here and in similar student errors. However, to direct and support meaningful work among high school students with instrumented calculus techniques, teachers must certainly have overcome such challenges themselves.

Challenge 4: theoretical work with calculus

The main topics in calculus – such as limits, derivatives and integrals – involve two groups of praxeologies: algebraic praxeologies (based on methods and rules for calculating limits, derivatives etc.) and topological praxeologies (based on existence problems and definitions of limits, derivatives etc.). It is a common trait of secondary level calculus in several countries, including Denmark, to treat mainly the algebraic praxeologies, with heavy use of instrumented techniques. From the point of view of university mathematics, this means that students work mainly consists in “finding” objects (such as limits) for which they have no formal definition or criteria of existence. We refer to Barbé et al. (2005) and Winsløw (to appear) for a more detailed discussion of this point.

A problem on integrals

Integration is perhaps the most advanced topic that is dealt with in Danish high school mathematics. The praxeologies *o* taught there generally involve informal explanations of what the definite integral computes (certain areas, averages etc.) and how to compute it (techniques based on antiderivatives or push button use of software); at the most advanced levels, we also find sketchy arguments of the fundamental theorem of calculus, bypassing any serious criteria for the existence of the integral (for instance, a certain “area function” corresponding to a given function f is assumed to exist, and then it is shown informally that its derivative is f ; see Winsløw, to appear, for more details).

At the same time, some Danish textbooks develop a more theoretical approach to integrals as an optional complement to the calculation oriented core material; but teachers are unlikely to use these and other opportunities for relating the integral more substantially with geometry (in particular area) if they have a less than familiar relationship with the subtleties involved in defining, rather than just computing, the integral of a function.

At the university, the students are in fact given rigorous definitions of integrals (in fact, various alternatives), as part of a more or less comprehensive treatment of real analysis. The question, naturally, is what students in position σ retain from these presentations and if they are able to make use of $R_U(\sigma, \omega)$ to build a new relationship $R_U(\sigma, o)$ which is relevant to the teaching of o in high schools.

A minimal interpretation of Klein's ideal of teaching "from a higher standpoint" is that students are able to use and reason about theoretical statements on integrals in a mathematically sound way, so that they will not simply be *forced* to resort to unreasoned statements once they become teachers. For instance, they should be able to explain how the Riemann integral gives sense to the area of certain subsets of the plane, and they should be able to engage in meaningful reasoning about basic notions in o such as integral, integrability, antiderivative and continuity. As we shall see, this is far from guaranteed by the fact that all of this (and much more!) has been presented to students in undergraduate courses.

Here is an example of a task given to the students in UvMat in view of developing $R_U(\sigma, \omega)$; in the context, it was clear that "integrable" is to be understood in the sense of Darboux:

Let the function f be integrable on the interval $I = [0, 1]$, and define the function F by

$$F(x) = \int_0^x f(t) dt, \quad x \in I.$$

Justify with rigorous reasoning whether or not it follows from what is given that F is continuous.

Justify, likewise with rigorous reasoning, whether or not one can deduce that F is differentiable.

The questions are open ended in the sense that they ask not just for a proof, but also for an answer to be proved. The first question requires a simple application of the boundedness of f on $[0, 1]$, while the second question requires a (non-continuous) counterexample such as the indicator function of $[0, \frac{1}{2}]$.

None of the students gave a fully satisfactory solution, while about 1/3 came close. Two types of shortcomings were prevalent among the rest. One is to present a sequence of statements which involve relevant notions and locally appear sound, but with little logical coherence among the statements. The following gives an impression of what that kind of "reasoning" may look like in an attempt to prove continuity of F (we stress that we copied and translated the student production

exactly, including what appears to be accidental errors or pretended reasoning):

... Let $\varepsilon > 0$. We look at: Let $x_0 \in I$. $|F(x_0) - F(x)| = \int_x^{x_0} f(t) dt$,

by the subdivision rule. $\int_x^{x_0} f(t) dt \rightarrow 0$ as $x_0 \rightarrow 0$.

Choose δ so that $|x_0 - x| < \delta \Rightarrow \int_x^{x_0} f(t) dt < \varepsilon$

so that it is concluded that F is continuous ...

The other shortcoming is the widespread mistake to consider the integral $F(x) = \int_c^x f(t) dt$ to be *defined* as the value at x of the *antiderivative* of f , which satisfies $F(c) = 0$ (this turns the Fundamental Theorem of Analysis into a circular statement in the context, where we recall that the definition is that due to Darboux). Students produce disguised versions of this, as in the following (erroneous) student answer to the second question:

By the definition of antiderivative we get that if $F(x) = \int_c^x f(t) dt$, $x \in I$, then F is differentiable in each point $x \in I$ with derivative $F'(x) = f(x)$.

This kind of “solution” both reflects a resisting pseudo-definition (or explanation) remembered from secondary school, where the only explicit formula for the integral is the one involving an antiderivative of the integrand, and an insufficient experience of students with theoretical work linked to the notions of integrability (in the sense of Darboux) and continuity. Such basic shortcomings are difficult to deal with in a capstone course where one assumes the basics of the university programme as a starting point for developing the “higher viewpoint” on elementary mathematics.

Construction of the exponential function

An obvious and potentially rich topic in a capstone course like UvMat is to study and indeed define the meaning of the exponential expression a^x for $a > 0$ and, crucially, with x an arbitrary real number. In high school, such expressions are introduced very early as examples of elementary functions, which will later become central examples and building blocks in the study of calculus. The challenge for the high school teacher is to give some meaning to a^x in the absence of any rigorous theory of real numbers and previous to the study of limits and other elements of the calculus. The common approach is to give a more or less detailed algebraic justification of the

formula $a^{m/n} = \sqrt[n]{a^m}$, and then merely claim that one can extend this definition from rational to arbitrary real numbers. Here are some explanations of this extension in Danish high school textbooks:

The power is calculated by approximating the exponent by a finite decimal number. How many decimals you include depend on the required accuracy (Timm & Svendsen, 2005, 26; translated from Danish by the authors)

In Chapter 3 we saw how to calculate powers where the exponent is integer and positive, 0, integer and negative, and rational (fraction). Strictly speaking we have not explained the meaning of a symbol like $7^{\sqrt{3}}$ but we assume CAS will take care of this. (Carstensen, Frandsen & Studsgaard, 2006, 82; translated from Danish by the authors)

For the following discussion, we denote by ω the mathematical praxeology built from the university experience, in view of establishing exponential functions a^x and their basic properties, with a complete mathematical theory to justify the extension to real numbers. In the high school version o of ω , the exponential functions will certainly have to be defined on the domain of real numbers; but at the moment where they are introduced, the available theory excludes a rigorous theoretical justification, so that any justification will need to be somewhat informal, as the above examples suggest. Still, the quality of explanation and activities proposed by teachers could certainly vary a lot, from the worst (not recognizing the problem or believing it to be insignificant) to the better (proposing a range of activities and explanations which could even anticipate or prepare a more rigorous work with approximations and limits of functions).

University students' relationship to the theoretical level of ω can be surprisingly weak. When informally asked, our students generally affirm that they have never seen a rigorous definition of a^x and in fact never considered that as a problem. In fact, Klein (1908, pp. 144-162) criticised the "algebraic" (and incomplete) approach to exponentiation which we just outlined, and instead proposed a "modern" approach based on logarithms and complex functions. This is, however, quite far from any feasible first approach in high school.

In UvMat, students instead work to close the gaps of the "algebraic approach": this approach is at least partially accessible to high school students, and with a strong familiarity with the gap (and what is needed to fill it), teachers are likely to make more meaningful transpositions, adapted to the capacities of their students.

This "gap closing" work essentially amounts to formulate and prove the following lemma (with free use of theoretical results they

have encountered in real analysis, and taking into account the algebraic definition of a^x for rational x):

Lemma. If a is a strictly positive real number, x is a real number and (r_n) is a sequence of rational numbers converging to x , then the sequence (a^{r_n}) is convergent. Moreover if (q_n) is another sequence of rational numbers converging to x , then the limit of the sequence (a^{q_n}) is the same as the limit of (a^{r_n}) .

Any proof will have to invoke deep properties of the real numbers, in particular they will have to make use of completeness in some form. We omit the details here and refer interested readers to Bremigan, Bremigan & Lorch (2011, pp. 294-295) for a possible proof. Given that familiarity is sought, we do not simply show students such a proof, but instead let them construct key parts of it. In fact, the students' lack of experience with theoretical work in this context - experienced in previous editions of the course - led us to propose in which students had to carry out smaller steps of the proof.

Passing now to observations from student work, we wish to emphasize a main challenge for students which can be resumed as follows: *the knowledge that they are supposed to use is not always what is most familiar to them*. For instance, one task given to students was to prove, through a number of given steps, the following special case of the above lemma: *If a is a strictly positive real number, and if (r_n) is any sequence of rational numbers converging to 0, then the sequence (a^{r_n}) converges to 1*. Several students at some point invoked properties of exponential functions, such as continuity (then, the proof is of course trivial).

One interpretation of this is that we had underestimated the need of making explicit what theoretical elements it makes sense to use when the task is to construct the function from scratch. On the other hand, it may not surprise that some students found it weird that they *could and should* use (for them) more advanced theoretical knowledge related to order and convergence in \mathbb{R} , but were criticized for using familiar results like the continuity of exponential functions.

The students' lack of experience with theory building had led us to having them work only on selected, local parts of a proof, but this appeared clearly insufficient to give them the "higher viewpoint" on the supposedly familiar and elementary object (the exponential function). Solving this challenge for capstone courses with aims like those of UvMat may require that *students are given more and earlier experiences at university with active construction of larger theoretical structures, involving definitions, partial results, proofs etc.*

Challenge 5: autonomous study beyond textbooks

To have the students search and use mathematical text by themselves, in view of solving a mathematical problem related to a high school mathematical praxeology o , can be seen as one of the most advanced and difficult relationships of type $R_U(\sigma, o)$ which one may seek to develop in a capstone course.

To exemplify this, consider the technique of least squares for simple linear regression – regularly taught in Danish high school. The type of task generating o consists in determining, for a data set $\{(x_1, y_1), \dots, (x_n, y_n)\} \subseteq \mathbb{R}^2$, the line $y = ax + b$, which approximates the data points best in the sense that the square sum $S(a, b) = \sum_{k=1}^n (ax_k + b - y_k)^2$ is minimized. The technique is just a formula (often implicit in “push button” instrumented technique); but how to justify it in a way accessible to high school students?

Notice first that relevant university level theory blocks could come from both calculus and linear algebra parts of the undergraduate curriculum. Many university textbooks use calculus, and here one easily finds that S has a unique critical point; it takes more theoretical reasoning to actually prove that this point is a minimum. A similar application of linear algebra can be obtained using the orthogonal projection of \vec{y} onto the two-dimensional subspace of \mathbb{R}^n spanned by $\vec{x} = (x_1, \dots, x_n)$ and $\vec{1} = (1, \dots, 1)$; the solution is simply the coordinates of that projection in the basis $\{\vec{x}, \vec{1}\}$. But high school theory blocks do not include partial derivative tests or orthogonal projections in n -dimensional Euclidean space; so although these arguments are accessible to university students, they may not really use them in a high school context, even in somehow simplified forms. Alternatives are not found in Danish high school text books, and they are not (as of early 2013) so easily found on the Internet; but a slightly more insistent Google search does lead one to texts like Key (2005), with an elementary proof based on “completion of squares” (that is, a slightly more complicated use of arguments which are, in principle, familiar to high school students, from the setting of quadratic equations).

During the 2011 edition of the course, we asked students to autonomously search for a proof, which is within the reach of high school mathematics (i.e. common theory blocks of a suitable o). This was impossible for all students, even with some help for getting started on the search; instead they all attempted to elementarise the calculus proof as an “analogue” of a one variable optimization problem.

In fact, students get very little experience and aptitude from their undergraduate studies when it comes to autonomous search and study of mathematical literature, in view of solving a concrete problem (such as finding an alternative proof, getting ideas for posing exercises etc.). Finding ways to change this aspect of students' relationship to mathematical praxeologies in this direction appears to us a main open problem in teaching this course. The problem is not only that they have relatively little experience with autonomous search for resources, especially with constraints like "find a solution accessible for high school students" (as in the case above). The difficulty lies also, it seems, in distinguishing potentially useful resources from irrelevant ones, and especially in working with the first type of resources as one rarely find a complete "solution" in one resource. In the case above, even after finding a text like Key (2005), details need to be worked out by the students to realize whether or not one has found what one was looking for –and then to work out the details to give an explicit and personalized solution, as commonly required in mathematics courses. It should be mentioned that in 2012, we simply gave the text by Key to students, and most of them were then able to accomplish that last step.

Based on this and similar experience, we believe that a development of students' relationship to mathematical praxeologies to include capacities for autonomous search and use of mathematical literature will be a difficult challenge for any capstone course departing from standard, textbook based bachelor programmes in pure mathematics. We also think that it is one of the most important aims in capstone courses for future teachers, given that teachers can – sometimes must – work with a wide variety of resources (cf. Gueudet & Trouche, 2009) – which nowadays naturally include Internet based ones. This aim is particularly evident in view of a context like Danish high school where significant parts of the teaching are done as supervision of individualized and multidisciplinary student "projects".

5. CONCLUSIONS AND PERSPECTIVES

In this paper, as partial answers to RQ1 and RQ2, we have uncovered a number of requirements and obstacles to construct a "bridge" between the mathematical praxeologies of contemporary university and high school, in the context of a capstone course where high school level calculus is studied and put into perspective using technical and theoretical elements of the university programme. Certainly the basic "double discontinuity" identified by Klein remains for the students whose academic preparation for high school teaching is mainly based

on university mathematics studies. In fact, the institutional and societal conditions have clearly changed considerably in 100 years, in ways that tend to widen the gap to be bridged, as explained in Sec. 2. On this account, it should be noted that it is no longer assumed that strong mathematical preparation suffices alone to be a professional mathematics teacher. On the other hand, study components like capstone courses (and other study units with a didactical focus) typically involve working with a multiplicity of techniques and theoretical perspectives on high school mathematical **praxeologies** - which do require a strong mathematical preparation, but perhaps with changed emphases relative to current practices in mathematics programmes.

Klein's vision of an "advanced viewpoint" is potentially relevant to develop teachers' "deep" knowledge of the high school subject. But this vision is no easier to realise today than in Klein's time: **the challenges presented** above point especially to the shortcomings of students' grasp of relevant university mathematics, and to the difficulty of creating situations where they can experience and realize the relevance of what they do know for solving problems related to calculus as taught in high school.

As regards **calculus praxeologies** taught in high school, the increasing importance of instrumented **techniques**, as well as the informal nature of **theoretical blocks**, requires specific attention to the future high school teachers' preparation at university: they must be prepared to solve and construct **tasks** for their students beyond a sequence of unrelated procedures which in the end amount to choosing relevant commands or buttons on a CAS device. They must, in particular, know a number of alternative approaches to the "hard" topological problems that, in calculus, are based on the completeness of real numbers - such as the definition and existence of central objects like elementary classes of functions³, limits, derivatives and integrals (**considered particularly in the context of challenge 2 and 4**). Some of these approaches could make use of the potential of CAS to visualize and compute (**Challenge 3**), while others must be based on simplified heuristic arguments and shortcuts (**for instance, to propose a manageable approach to exponential functions, as considered in the second part of Challenge 4**). Teachers knowing the "full story" are less likely to give their students the impression that mathematics is

³ The case of exponential and power functions was considered within "Challenge 4"; similarly, a rigorous construction of the trigonometric functions could be studied.

essentially a collection of useful procedures whose foundations remain *intrinsically* impossible to explain.

At the same time, and in a sense before that, we must also face RQ3, as we identified a number of shortcomings in some university students' relationship to mathematical practices which are or ought to be central also in their previous studies, such as the capacity to solve equations correctly (Challenge 1), to make use of calculus techniques in non-standard problems (Challenge 2), to make use of instrumented techniques to investigate concrete problems (Challenge 3), to devise simple but correct mathematical arguments (Challenge 4), and to search and study mathematical literature in an autonomous way, for instance to identify alternative proofs under boundary conditions for the machinery to be deployed (Challenge 5). In fact, all of these capacities need to be established in analysis and algebra courses before "visits" to advanced topics more distant to high school calculus. And even when that is done, so that "remedial measures" become less needed, Klein's problem remains and motivates a continuing effort to develop the contents and methods of capstone courses.

APPENDIX : DETAILED EXAMPLE OF ITEM ANALYSIS

The first question in the exam item discussed in "Challenge 1" is the following :

Solve the equation $\sqrt{2x+12}-1=x$; provide all intermediate steps of your solution.

Our coding of answers is based on the degree to which seven subtasks (given below) were identified and solved, and also on the explicit connections provided between them and the given task. We stress that there is no contention that these subtasks should be considered in a particular order or that all of them need to be addressed:

Subtask 1: Decide for what values of x the equation makes sense (namely $x \geq -6$)

Subtask 2: Rewrite the equation as $\sqrt{2x+12} = x+1$

Subtask 3: Infer that the result from task 2 *implies* that $x \geq -1$,

Subtask 4: Infer that the result from subtask 2 *implies* that $2x+12 = (x+1)^2$

Subtask 5: Observe that $x \geq -1$ and $2x+12=(x+1)^2$ together imply $\sqrt{2x+12}=x+1$ so that the given equation is logically equivalent to $2x+12=(x+1)^2 \wedge x \geq -1$

Subtask 6: Solve $2x+12=(x+1)^2$ (solutions: $x=\pm\sqrt{11}$)

Subtask 7: Identify the complete solution to the equation as $x=\sqrt{11}$.

We then created a table with one row for each respondent and one column for each subtask; the presence and character of solutions to the subtasks was then indicated, for each student, using a specific coding system.

It is important to note that our analysis of items was not “fixed” but could be changed to admit alternative, correct solutions. But for almost all items, we did in fact manage to predict the steps taken by students in the sense that they could be described as solving some subset of the subtasks identified. In the present case, this does not mean that they solved all subtasks or did them in the given order. In fact, a common technique was to solve subtask 2, 4 and 6 in that order, and then either (incorrectly) state $x=\pm\sqrt{11}$ as the solution, or insert the original equation and observe that only $x=\sqrt{11}$ “works”, so that this is the solution. This final step was coded as a partial solution of subtask 3 as it leads to observe that one concrete number less than -1 does not satisfy the equation; with the other steps, and of course an explicit argument, this is indeed a valid solution. We note also that significantly different algebraic techniques, such as using the identity

$$(\sqrt{2x+12}-(x+1))(\sqrt{2x+12}+(x+1))=2x+12-(x+1)^2$$

are indeed possible *a priori*; but techniques which did not appear in student solutions were not included in our model for coding them.

Clearly, the assessment of the task is not completed by using this coding, which amounts merely to identifying the students’ *technique*; their *technology* and *theory* is then identified through explicit connections between the steps (as indicated in the main text on Challenge 1), explicit appeal to specific results, erroneous inferences, etc. For instance, some explanation of the logical validity of “all intermediate steps” is needed, even if one proceeds by implication and then tests the possible solutions by insertion in the end. On the other hand, the table resulting from the coding related to subtasks was indeed very useful to provide an overview of students’ capacities and challenges related to a given task.

Remerciements. Nous tenons à remercier les trois rapporteurs et l'éditeur de cet article, qui ont fourni des critiques et des suggestions lucides grâce auxquelles notre texte a pu être amélioré sur de nombreux points, et qui confirment pour nous la valeur du fait qu'une communauté de recherche partage des références communes qui développent de façon cumulative.

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